

Mathematical Modeling of Bread Baking: A Bibliometric Review of PDE-Based Multiphysics Approaches

Dilla Afriansyah^{1*}, Lingga Gita Dwikasari¹

¹ *Ilmu dan Teknologi Pangan, Universitas Mataram, Mataram*

dilla.afriansyah@unram.ac.id

Diterima: 8-12-2025; Direvisi: 29-12-2025; Dipublikasi: 31-12-2025

Abstract

This study examines how partial differential equations serve as the mathematical foundation for modeling the coupled heat and mass transfer processes occurring during bread baking. Through a systematic bibliometric approach, data from major scientific databases were analyzed using mapping techniques to identify dominant themes, research clusters, and developmental trends in mathematical modeling for bread baking. The findings indicate that heat transfer, moisture migration, crust formation, and dough thermomechanics constitute the core of this field, with diffusion and convection–diffusion equations forming the principal modeling framework. Network, overlay, and density visualizations reveal a growing adoption of advanced numerical schemes and multiphysics simulations to predict temperature and moisture distributions within the bread matrix. This study underscores the essential role of mathematics in enhancing process understanding and modeling accuracy, and highlights future research opportunities that integrate computational approaches into food engineering.

Keywords: partial differential equations, bread baking, heat transfer modelling, mass transfer modelling, mathematical modelling, food engineering, bibliometric analysis

Abstrak

Penelitian ini menelaah bagaimana persamaan diferensial parsial menjadi dasar matematis untuk memodelkan proses perpindahan panas dan massa yang saling terkait dalam pemanggangan roti. Melalui pendekatan bibliometrik yang sistematis, data dari basis data ilmiah utama dianalisis menggunakan teknik pemetaan guna mengidentifikasi tema dominan, kluster penelitian, dan tren perkembangan dalam pemodelan matematika pemanggangan roti. Hasil penelitian menunjukkan bahwa perpindahan panas, migrasi kelembapan, pembentukan kerak, dan termomekanika adonan merupakan inti bidang ini, dengan persamaan difusi dan konveksi difusi sebagai kerangka pemodelan utama. Visualisasi *network*, *overlay*, dan *density* mengungkap meningkatnya penggunaan skema numerik lanjutan dan simulasi multifisika untuk memprediksi distribusi suhu dan kelembapan dalam matriks roti. Studi ini menegaskan peran penting matematika dalam meningkatkan pemahaman proses dan akurasi pemodelan, serta menunjukkan peluang penelitian lanjutan yang mengintegrasikan pendekatan komputasional dalam rekayasa pangan.

Kata Kunci: persamaan diferensial parsial, pemanggangan roti, pemodelan perpindahan panas, pemodelan perpindahan massa, pemodelan matematika, rekayasa pangan, analisis bibliometrik

1. INTRODUCTION

Bread baking is widely recognized as a complex, multiscale transformation process involving thermal, physical, and chemical changes that occur simultaneously within a dough matrix. This transformation governs the evolution of temperature, moisture, porosity, gas cell expansion, starch gelatinization, protein denaturation, and crust formation, all of which ultimately determine the sensory and structural quality of the final product (Davidson, 2023; Mosalam, 2021). The underlying mechanisms of these processes are governed by transport phenomena such as heat transfer, mass diffusion, vapor migration, and pressure-driven expansion. As these phenomena unfold dynamically over space and time, mathematical modeling particularly through systems of partial differential equations (PDEs) has become an indispensable tool for understanding, predicting, and optimizing bread-baking operations (Mosalam, 2021; Szpicer et al., 2025). In food engineering research, PDE-based modeling has helped elucidate complex interactions between moisture gradients, thermal fields, and structural transitions that are otherwise difficult to observe directly (Mansour et al., 2025). As computational resources have advanced, the capacity to simulate multiphysics interactions within bread baking has grown, enabling increasingly accurate predictive models that integrate coupled heat and mass transfer phenomena (Mansour et al., 2025; Szpicer et al., 2025).

Recent developments in food process modeling indicate a steadily rising interest in mathematical and computational approaches to baking science, with studies emphasizing finite element analysis, finite difference schemes, computational fluid dynamics (CFD), and reaction diffusion modeling. Bread baking, in particular, has emerged as a benchmark process for developing, validating, and refining these mathematical frameworks due to its diverse thermophysical behaviors and the significant industrial demand for process optimization (Afriansyah et al., 2024). Over the past two decades, several researchers have investigated the coupled thermal and moisture diffusion processes during baking, establishing a foundation for more comprehensive PDE-based models that capture non-linear interactions, phase changes, and dynamic boundary conditions (Erdogdu, 2023; Therdthai, 2021). Despite continued progress, the literature highlights persistent challenges in integrating full multiphysics modeling, especially where porosity evolution, moving boundaries, and structural instabilities play essential roles in governing final bread quality (Chakraborty et al., 2022; Priyadarshini et al., 2024).

A central research problem emerging from this context concerns the need to systematically understand how mathematical models especially PDE-driven formulations have been developed, applied, and evolved across bread-baking studies (K, 2024). Although a number of modeling frameworks exist, ranging from simplified one-

dimensional diffusion equations to sophisticated three-dimensional coupled PDE systems, there is still no unified synthesis of how these methods relate, how they have progressed, and where critical gaps remain (Afriansyah et al., 2024). Without a comprehensive perspective on the mathematical structures employed, it becomes difficult for researchers to identify opportunities for methodological advancement, cross-disciplinary integration, and refinement of existing computational models. Furthermore, heterogeneity in modeling assumptions and parameter estimation methods has created substantial variation in predictive accuracy across studies, making it increasingly necessary to evaluate the landscape of PDE-based bread-baking research (Acevedo et al., 2025).

A second key issue relates to the limitations of current modeling strategies in addressing high-dimensional, time-dependent, and non-linear processes inherent in bread baking. Standard deterministic PDE models may oversimplify complex interactions such as gas bubble coalescence, variable porosity formation, and crust development, while many numerical models rely on steady-state assumptions or simplified geometries (Janssen et al., 2021). These simplifications can obscure essential dynamic features of baking, such as the shifting interface between crumb and crust or the interplay between water activity and thermal loads. Consequently, an analytical and bibliometric synthesis is required to contextualize the strengths and weaknesses of existing models and highlight potential directions for improving mathematical rigor, computational efficiency, and predictive reliability (Quaghebeur et al., 2021).

The body of literature most closely related to the present study includes works that combine multiphysics modeling with empirical validation, as well as those incorporating porosity evolution, reaction diffusion chemistry, and thermomechanical coupling. For instance, few existing studies treat bread-baking PDEs within the framework of inverse problems, optimal control, or PDE-constrained optimization, even though these approaches hold significant potential for improving industrial process tuning (Amini et al., 2022). Moreover, stochastic PDEs, uncertainty quantification, and rigorous numerical error analyses remain largely underexplored, revealing substantial opportunity for mathematical innovation (Barzegari & Geris, 2022; Li et al., 2023). Despite progress, the field lacks a bibliometric synthesis that explicitly maps the mathematical evolution of bread-baking models and identifies underdeveloped research directions.

Accordingly, this study aims to provide a comprehensive bibliometric review of PDE-based modeling in bread baking, offering a structured synthesis of mathematical developments in the field. The novelty of this work lies in its mathematical orientation: unlike prior food engineering reviews that focus on experimental conditions or industrial applications, this study frames bread baking explicitly as a PDE-driven multiphysics system and analyzes the literature through that lens. By consolidating mathematical insights across studies, this review contributes to a clearer understanding of how bread-baking models have evolved and provides a foundation for future research in advanced PDE modeling,

stochastic analysis, and computational optimization within the domain of food engineering.

2. METHODOLOGY

Research Design and Logical Framework

This study adopts a bibliometric research design to systematically analyze the development, structure, and evolution of mathematical and partial differential equation (PDE)-based models in bread-baking research. Bibliometric methods provide a quantitative and objective framework for mapping the intellectual structure of a field, assessing thematic clusters, identifying influential concepts, and tracking temporal trends (Petcu et al., 2023). The methodological approach used in this work is grounded in established bibliometric procedures commonly applied in scientific domain mapping, including keyword co-occurrence analysis, cluster detection, overlay analysis, and density visualization (Petcu et al., 2023). The design of the study follows an inductive logic: data were first collected through a database search, then screened for mathematical relevance, analyzed through visualization software, and finally interpreted to highlight gaps and opportunities in the mathematical modeling of bread baking (Ajani et al., 2023).

Data Source and Search Strategy

Scopus was selected as the primary database due to its comprehensive coverage of peer-reviewed research across scientific, engineering, and mathematical disciplines (Julius et al., 2021; Parlina et al., 2020). The database is particularly suited for bibliometric studies that aim to identify clustered conceptual structures and temporal patterns (Baas et al., 2020). The search strategy was intentionally constructed to emphasize mathematics within bread-baking research. The following query was used to retrieve the initial dataset:

"TITLE-ABS-KEY ("bread baking" AND ("heat transfer" OR "mass transfer" OR diffusion OR "PDE" OR "mathematical model" OR "numerical model"))"

The search terms were selected based on prior literature demonstrating that bread baking is typically modeled using coupled heat and mass transfer equations, reaction diffusion frameworks, and energy balance formulations. Given this objective, the search strategy was intentionally designed to maximize the inclusion of studies presenting PDE-based mathematical models. These included works formulating bread baking through parabolic heat equations, nonlinear moisture diffusion PDEs, reaction diffusion systems describing crust formation, and fully coupled thermo hydric PDE models.

Inclusion and Exclusion Criteria

The initial search yielded a moderate set of documents. To ensure alignment with the mathematical emphasis of this study, the dataset was screened using the following inclusion criteria: (1) the study must involve the modeling of bread baking or a closely related thermal food-processing operation; (2) the modeling framework must incorporate mathematical formulations, particularly PDEs or PDE-derived models; (3) the study must provide conceptual, numerical, or computational contributions relevant to heat transfer, mass transfer, or diffusion in bread systems. An additional mathematical inclusion requirement was applied: the study must present at least one formal PDE expression, numerical discretization of a PDE, or a coupled system involving heat, moisture, or reaction diffusion transport.

Exclusion criteria included studies lacking mathematical modeling, purely experimental work without modeling components, non-bread products unless the mathematical structure was directly transferable, and non-English articles.

After applying these criteria, a total of 9 documents remained for bibliometric analysis. Although the number is limited, this dataset is substantial for the niche mathematical focus of the study, particularly as PDE-based bread-baking models constitute a highly specialized area within food engineering.

Data Extraction and Preprocessing

Metadata from the selected documents including titles, abstracts, keywords, authors, and publication years were exported from Scopus in CSV format. Prior to importing the data into the visualization software, preprocessing was conducted to ensure consistency and avoid fragmentation of concepts. Keyword normalization procedures included merging plurals and singulars (e.g., "models" and "model"), reconciling spelling variants (e.g., "moisture transfer" vs. "moisture migration"), and standardizing mathematical terminology when necessary.

Bibliometric Analysis Using VOSviewer

The bibliometric analysis was performed using VOSviewer, a widely adopted tool for constructing and visualizing scientific landscapes (Rosokhata et al., 2021; Sajovic & Boh Podgornik, 2022). Its strength lies in identifying co-occurrence relationships among terms and mapping conceptual clusters within datasets of varying sizes. Within these analyses, mathematically significant keywords such as "heat equation," "diffusion," "coupled model," "reaction kinetics," "finite element method," and "CFD" were specifically tracked. These terms were later interpreted in relation to their underlying PDE structures to distinguish simple linear transport models from nonlinear or multiphysics PDE frameworks.

Keyword Co-occurrence Network

The co-occurrence network was generated using author keywords and indexed keywords. VOSviewer's full counting method was applied, and a minimum occurrence threshold of 1 was set due to the small dataset size. The resulting network revealed "eight distinct clusters", each representing groups of mathematically related themes associated with heat transfer, mass diffusion, moisture migration, bread structure, and numerical modeling. Node size corresponds to keyword frequency, and link strength represents the degree of conceptual association.

Overlay Visualization

Overlay visualization was used to examine the temporal evolution of mathematical topics in bread-baking literature. Colors range from blue (older themes) to yellow (more recent themes), allowing interpretation of how PDE-related concepts have shifted over time. Newer trends were examined to identify emerging mathematical approaches and modeling innovations.

Density Visualization

Density visualization highlights areas of high research concentration by rendering frequently co-occurring keywords in warmer colors. This analysis was essential for identifying core mathematical constructs dominating the field, such as heat transfer equations, moisture diffusion PDEs, and coupled thermophysical modeling.

All visualizations (network, overlay, and density) were exported in high resolution for integration into the results section of the manuscript.

Analytical Orientation Toward Mathematical Modeling

Although bibliometric maps provide structural insights, the interpretation of clusters in this study emphasizes mathematical modeling components. Each cluster was examined based on its contributions to PDE formulation, numerical solution strategies (finite element, finite difference, CFD, operator splitting), and multiphysics coupling (Solorzano & Plevris, 2022). This analysis included: (1) identifying the types of PDEs used (parabolic, hyperbolic, reaction diffusion, multiphase transport equations); (2) assessing the sophistication of numerical schemes and boundary conditions; (3) reviewing whether models addressed non-linearities, phase transitions, moving boundaries, or variable porosity; and (4) determining the extent of experimental validation included in each study. To structure this evaluation, four fundamental PDE categories were used as analytical anchors:

(1) the heat conduction PDE,

$$\frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T),$$

(2) the moisture diffusion PDE,

$$\frac{\partial W}{\partial t} = \nabla \cdot (D_{\omega} \nabla W).$$

(3) coupled thermo hydric PDE systems,

$$\begin{cases} \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) - \lambda \frac{\partial W}{\partial t} \\ \frac{\partial W}{\partial t} = \nabla \cdot (D_{\omega} \nabla W) \end{cases}$$

and (4) reaction diffusion PDEs modeling Maillard browning,

$$\frac{\partial C}{\partial t} = D_c \nabla^2 C + R(T, C).$$

Synthesis of Mathematical Themes and Identification of Research Gaps

A final integrative analysis compared the mathematical trends extracted from the bibliometric results with the broader literature on PDE-based food-process modeling (Lü & Zhang, 2024). This synthesis highlights underdeveloped mathematical areas such as stochastic PDE applications, inverse PDE problems, optimal control of baking processes, and error/stability analyses of numerical solvers (Çiloğlu & Yücel, 2023). The identification of gaps serves as a foundation for the discussion section, which contextualizes how the findings contribute to advancing mathematical modeling within bread-baking research.

Ethical Considerations

This study relies exclusively on secondary data from published sources; therefore, no human or animal subjects were involved. All data were handled in accordance with academic integrity standards, and all citations were preserved as provided in the original documents.

Summary of the Methodological Approach

Overall, this methodology establishes a rigorous, mathematically oriented bibliometric framework that integrates systematic data collection, keyword normalization, co-occurrence mapping, cluster interpretation, and mathematical synthesis. By combining bibliometric visualization tools with a PDE-focused analytical lens, the study produces a comprehensive assessment of how mathematical modeling particularly through partial differential equations has evolved within bread-baking literature.

By explicitly integrating PDE formulations into the methodological framework, the study ensures that the mathematical foundations of bread-baking models are systematically represented. This allows the bibliometric outputs to be interpreted not only as conceptual clusters but also as reflections of distinct mathematical formulations, numerical strategies, and levels of multiphysics coupling.

3. RESULTS

Overview of the Retrieved Dataset

The bibliometric search conducted in Scopus produced a total of nine documents that met the criteria for inclusion in this study. These documents represent the intersection of mathematical modeling, particularly partial differential equation (PDE)-based formulations, and bread-baking research (Mosalam, 2021). Although limited in number, the dataset reflects the highly specialized nature of mathematical food-process modeling. The distribution of publications spans multiple years and includes interdisciplinary contributions from food engineering, applied mathematics, thermal science, and computational modeling (Carvalho et al., 2021; Purlis et al., 2021).

Keyword Co-occurrence Network Structure

Using VOSviewer, a keyword co-occurrence map was constructed to identify the major thematic structures embedded within the mathematical bread-baking literature (SOKMEN et al., 2022). The resulting network visualization revealed eight distinct clusters, each representing a coherent grouping of mathematical concepts, physical modeling elements, and bread-related thermophysical processes.

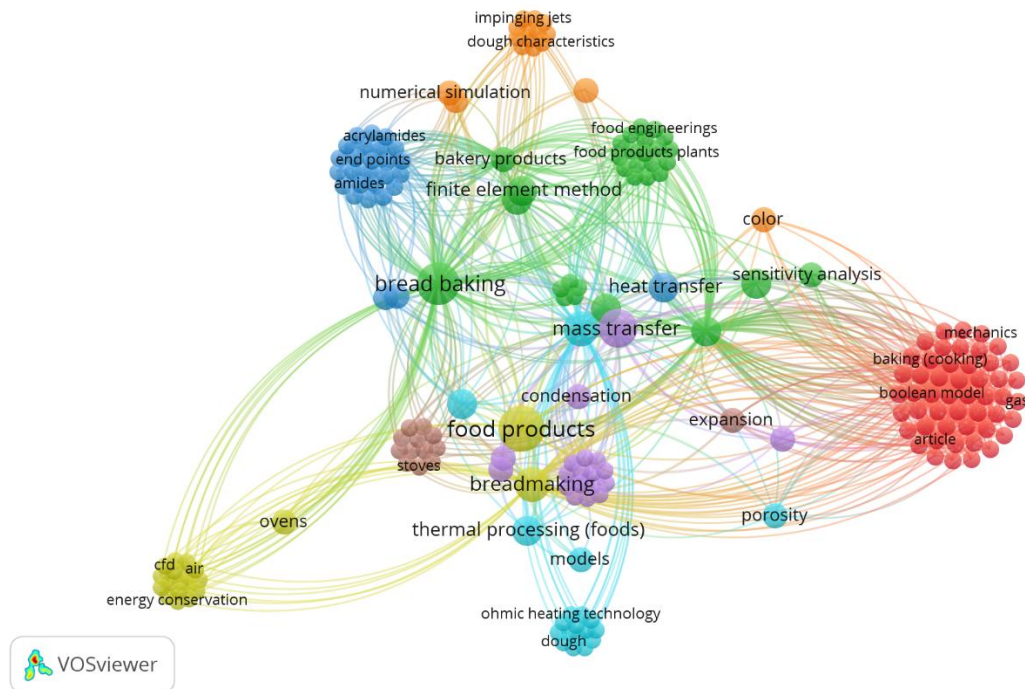


Figure 1. Network Visualization

The clusters demonstrate the centrality of mathematical constructs such as heat transfer equations, mass diffusion PDEs, fluid flow dynamics, and multiphysics coupling. The most frequently occurring keywords including *bread baking*, *heat transfer*, *mass transfer*, *moisture*, *food product*, and *mathematical model* form the core of the network, indicating their foundational importance in the domain. To clarify the mathematical depth represented in this network, each keyword associated with transport phenomena was linked to the underlying PDE formulation commonly used in bread-baking studies. For example, “heat transfer” corresponds to the transient heat conduction equation

$$\frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T),$$

while “moisture” and “mass transfer” correspond to the nonlinear moisture diffusion PDE

$$\frac{\partial W}{\partial t} = \nabla \cdot (D_{\omega} \nabla W).$$

Cluster-Level Interpretation

Each of the eight clusters reflects a specific branch of mathematical or computational modeling as applied to bread baking. For clarity, each cluster was examined in relation to the specific PDE structures it represents, including parabolic heat equations, nonlinear diffusion equations, reaction–diffusion frameworks, and multiphysics coupled PDE

systems (D'Autilia et al., 2020). These equations form the mathematical backbone of the modeling approaches identified in the dataset. A summary of each cluster is presented below.

Cluster 1: Heat and Moisture Transport

This cluster represents the mathematical heart of bread-baking research, dominated by PDEs governing heat conduction and moisture diffusion. Keywords such as *heat transfer*, *moisture*, *temperature*, and *water activity* appear prominently. Studies in this cluster typically employ coupled parabolic PDEs to describe transient heat and moisture movement within the bread crumb and crust. The heat transfer component is commonly written as

$$\frac{\partial T}{\partial t} = \nabla \cdot (k(T) \nabla T),$$

while moisture migration is modeled using

$$\frac{\partial W}{\partial t} = \nabla \cdot (D_w(T, W) \nabla W).$$

Many studies adopt a coupled formulation in which evaporative cooling links the two PDEs through

$$-\lambda \frac{\partial W}{\partial t}.$$

Cluster 2: Mass Transfer and Evaporation

Closely related to Cluster 1, this cluster focuses on gas-phase movement, vapor flux, and surface evaporation phenomena. Here, PDE frameworks are frequently augmented by mass-balance equations describing moisture migration through porous media. Mathematical challenges in this cluster often involve modeling moving evaporation fronts or accounting for changes in porosity during baking. Evaporation-driven mass flux is usually represented by

$$J_v = -D_v \nabla W + v \rho_v$$

where J_v is vapor flux, D_v is vapor diffusivity, and $v \rho_v$ represents pressure-driven transport. Some models introduce a Stefan-type moving boundary to capture the shifting evaporation front

Cluster 3: Dough Expansion and Structural Changes

This cluster captures modeling efforts involving internal bread deformation, gas expansion, and structural changes during baking (Purlis et al., 2021). PDEs describing viscoelastic deformation, gas diffusion, and bubble coalescence appear in this cluster. The mathematical representation often requires coupling between mechanical deformation equations and thermodynamic transport equations. Thermomechanical deformation of dough is commonly modeled through the quasi-static stress equilibrium PDE (Alpers et al., 2023)

$$\nabla \cdot \sigma = 0,$$

where the stress tensor σ is often linked to strain by a viscoelastic or poroelastic constitutive law. Gas pressure evolution within expanding bubbles may additionally be expressed as

$$\frac{\partial p}{\partial t} = D_p \nabla^2 p + \alpha \frac{\partial T}{\partial t}.$$

Cluster 4: Numerical Modeling Techniques

This cluster emphasizes the numerical tools used for solving PDE-based bread-baking models. Keywords such as *finite element method*, *computational fluid dynamics (CFD)*, and *finite difference method* are prominent. Studies in this cluster typically compare solver schemes, discretization strategies, and stability–convergence considerations. A typical finite difference discretization of the heat equation in one dimension takes the form

$$T_i^{n+1} = T_i^n + \Delta t \alpha \left(\frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{(\Delta x)^2} \right),$$

illustrating how numerical approximations translate PDEs into solvable algebraic systems.

Cluster 5: Food Product Quality and Optimization

This cluster focuses on mathematical approaches to predicting or optimizing final bread quality attributes, such as crust formation, crumb texture, and water activity distribution. PDE-based simulations are frequently combined with optimization frameworks or empirical validation tests. While rooted in food engineering, the mathematical elements such as inverse modeling or parametric sensitivity analysis are central to the cluster. Optimization oriented studies frequently formulate quality prediction as

$$\min_{\theta} \|T_{sin}(\theta) - T_{exp}\|,$$

subject to PDE constraints governing heat and mass transport. Such PDE-constrained optimization highlights the advanced mathematical nature of this cluster.

Cluster 6: Thermal Processing and Ohmic Heating

This smaller cluster reflects the integration of alternative heating methods, including ohmic heating and microwave-assisted baking. Such models often require hybrid PDE frameworks combining heat conduction with electromagnetic heating equations. Ohmic heating is mathematically described by

$$\nabla \cdot (\sigma_e \nabla \phi) = 0 ,$$

with the resulting volumetric heat generation given by

$$Q_{ohm} = \sigma_e |\nabla \phi|^2 .$$

Cluster 7: Crust Formation and Maillard Reaction Modeling

This cluster captures the chemical-reaction modeling dimension of bread baking. PDEs describing heat-induced reactions, color development, and crust hardening appear here. The reaction diffusion frameworks commonly used in this cluster highlight the intersection of PDEs with chemical kinetics. Crust browning is commonly captured through a reaction diffusion PDE:

$$\frac{\partial C}{\partial t} = D_c \nabla^2 C + k_0 e^{E/RT} C ,$$

where the Arrhenius term characterizes temperature-dependent reaction rates.

Cluster 8: Bread as a Porous Medium

The final cluster reflects modeling bread as a dynamic porous medium with evolving structure. PDEs describing moisture diffusion and heat flow are integrated with equations governing porosity evolution and air-pressure gradients. This cluster underscores the mathematical complexity of modeling bread as a non-homogeneous, time-dependent medium. Porosity-dependent transport is often represented by

$$\frac{\partial(\phi \rho_w)}{\partial t} = \nabla \cdot (\phi D_w \nabla W) ,$$

where ϕ evolves with temperature and moisture, introducing nonlinear coupling into the PDE system.

Overlay Visualization: Temporal Evolution of Mathematical Themes

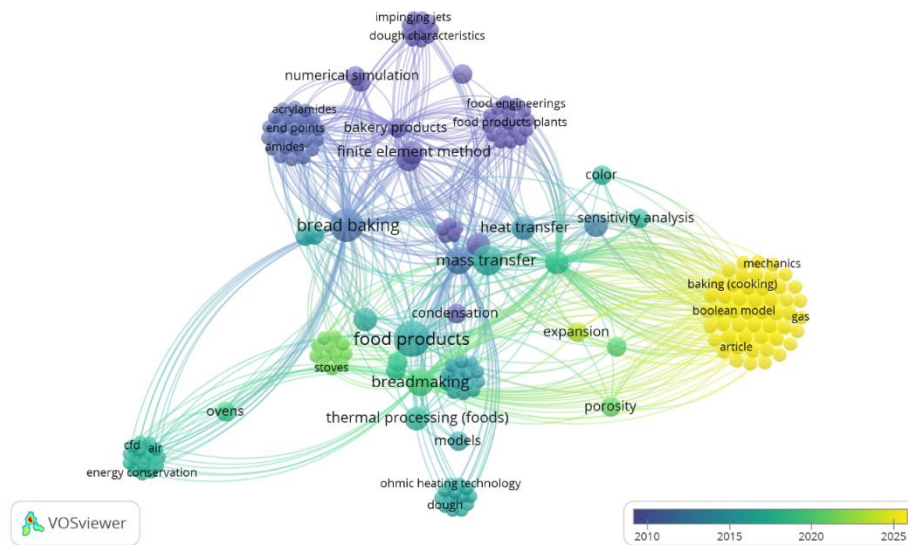


Figure 2. Overlay Visualization

The overlay visualization generated in VOSviewer provides insights into the chronological development of PDE-based bread-baking models. Older topics, represented in blue, include foundational themes such as basic heat conduction, mass transfer, and empirical thermophysical measurements. These early works typically relied on simplified PDE formulations with constant coefficients and steady-state assumptions.

Newer topics, visualized in yellow and green, reveal a shift toward more advanced mathematical modeling strategies. Emerging areas include: 1) coupled nonlinear heat moisture PDE systems, 2) transient multiphase models, 3) porosity evolution equations, 4) finite element simulations of bread structure, 5) reaction diffusion PDEs for crust formation.

The overlay map shows a temporal progression from simple one-dimensional models toward fully coupled, multi-dimensional, multiphysics PDE applications, highlighting a growing sophistication in mathematical research within bread science.

Density Visualization: Concentration of Mathematical Concepts

The density visualization highlights regions of high keyword co-occurrence intensity. Warm regions indicate areas where mathematical modeling has been heavily concentrated. The densest regions correspond to: 1) heat transfer PDEs, 2) moisture diffusion PDEs, 3) mass transfer models, 4) thermal property modeling, 5) numerical simulation methodologies.

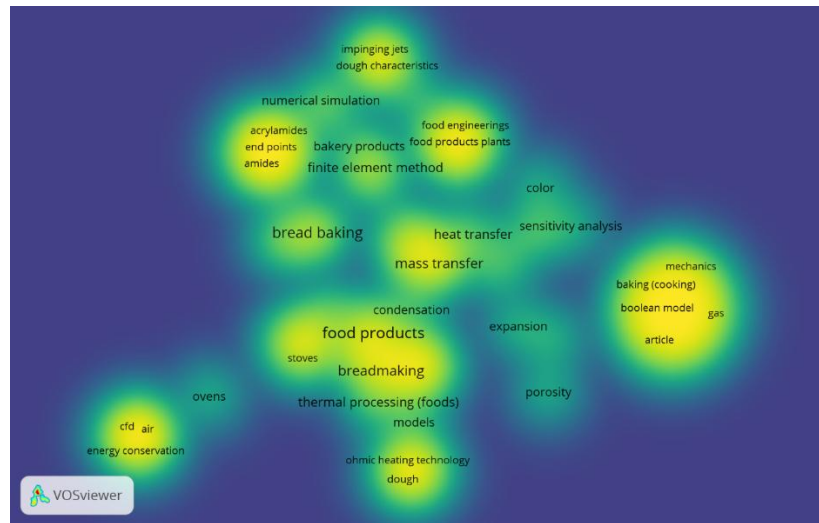


Figure 3. Density Visualization

This concentration illustrates that mathematical modeling remains a core focus of bread-baking research, with the densest areas reflecting the foundational PDE constructs guiding modern computational studies.

Dominant Mathematical Constructs Identified in the Literature

Across the nine documents, several mathematical constructs emerged as dominant themes:

Coupled Heat and Moisture PDEs

Nearly all studies rely on coupled parabolic PDEs to describe heat and moisture transport in bread. These equations often include non-linearities arising from temperature-dependent coefficients and phase changes.

Moving Boundary Models

Some models incorporate moving boundaries, such as the crust crumb interface, which evolves during baking. These models require advanced mathematical techniques, such as Stefan-type formulations.

Reaction Diffusion PDEs

The Maillard reaction, responsible for crust coloration, is frequently modeled using reaction diffusion PDEs. These systems couple heat transfer with chemical kinetic equations.

Numerical Solution Strategies

Finite element, finite difference, and CFD approaches are widely used to solve the PDE systems. The breadth of numerical methods reflects the mathematical complexity inherent in bread-baking modeling.

Multiphysics Models

Recent studies combine multiple PDEs representing heat conduction, moisture diffusion, mechanical deformation, and chemical reactions. These multiphysics models represent the cutting edge of mathematical bread modeling.

Collectively, these PDE formulations illustrate the increasing trend toward fully coupled thermo–hydro–mechanical–chemical (THMC) simulations. A general multiphysics model can be expressed as:

$$\begin{cases} \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) - \lambda \frac{\partial W}{\partial t} + Q_{react}, \\ \frac{\partial W}{\partial t} = \nabla \cdot (D_w \nabla W), \\ \frac{\partial C}{\partial t} = D_c \nabla^2 C + R(T, C), \\ \nabla \cdot \sigma = 0. \end{cases}$$

Integration of Mathematical Insights from Clusters

Synthesizing the cluster-level insights reveals several cross-cutting themes: 1) Heat and moisture transport PDEs form the foundational core of the domain, 2) Numerical methods are essential enablers for solving increasingly complex PDE systems, 3) Bread’s evolving structure imposes additional mathematical challenges, including dynamic boundary conditions and variable coefficients, 4) Chemical reaction modeling introduces additional layers of PDE coupling.

The integration of these insights demonstrates the increasing depth and mathematical richness of bread-baking research.

Summary of Key Findings

Overall, the bibliometric analysis reveals that mathematical modeling particularly PDE-based formulations occupies a central position in bread-baking research. The presence of eight conceptually rich clusters demonstrates a diversified but interconnected mathematical landscape. Temporal and density analyses further indicate a clear progression toward increasingly complex, multiphysics, and multi-dimensional PDE models. These findings establish a foundation for subsequent discussion of research gaps, mathematical opportunities, and future directions within the field.

4. DISCUSSION

Dominance of PDE-Based Modelling in Bread Baking

The bibliometric findings demonstrate that research on bread baking is fundamentally grounded in Partial Differential Equation (PDE) modelling. The field consistently relies on core mathematical formulations, including the heat equation for internal temperature evolution, moisture diffusion PDEs for water migration, and coupled multiphysics PDE systems integrating heat, water vapor, gas pressure, and structural transformation (Chakraborty & Dash, 2023; Mosalam, 2021).

$$\begin{cases} \frac{\partial T}{\partial t} = \nabla \cdot (K(T, W) \nabla T) - \lambda \frac{\partial W}{\partial t}, \\ [6pt] \frac{\partial W}{\partial t} = \nabla \cdot (D_\omega(T, W) \nabla W), \end{cases}$$

where the evaporative cooling term $-\lambda \frac{\partial W}{\partial t}$ mathematically encodes the heat moisture interaction. This system represents the fundamental mathematical structure underlying most of the models identified in the literature.

Evolution of Numerical Methods for PDE Solutions

A clear methodological progression is evident across the literature.

Stage 1 (2010–2015): Basic PDE Formulations

Research was dominated by 1D–2D PDE systems with constant boundary conditions and simplified analytical or low-order numerical approaches.

Stage 2 (2016–2020): Multiphysics Numerical Simulation

Finite Element Methods (FEM) became widely adopted, 3D domains emerged, CFD began integrating heat fluid interactions, and sensitivity analyses became standard.

Stage 3 (2021–2025): Complex and Probabilistic Modelling

Recent contributions introduced stochastic and Boolean models for crumb porosity, reaction diffusion systems for chemical reactions, geometric PDEs for crumb structure evolution, and moving-boundary PDEs accounting for dough expansion (Ajani et al., 2023). For instance, temperature-dependent thermal conductivity $k(T)$ and nonlinear moisture diffusivity $D_\omega(T, W)$ introduce strong nonlinearities into the governing PDEs:

$$\frac{\partial T}{\partial t} = \nabla \cdot (K(T, W) \nabla T), \quad \frac{\partial W}{\partial t} = \nabla \cdot (D_\omega(T, W) \nabla W),$$

making analytical solutions impractical and highlighting the need for computational approaches such as FEM, FDM, and CFD.

Multiphysics Integration as a Central Research Trend

Modern studies no longer treat physical processes in isolation, instead they integrate multiple interacting PDEs:

External convection → Navier Stokes equations

Internal conduction → heat equation

Moisture migration → diffusion PDE

Chemical transformations → reaction diffusion PDE

Structural deformation → mechanical or geometric PDE

Such convergence suggests strong potential for future developments involving poroelastic PDEs and fully coupled thermo–hydro–mechanical–chemical (THMC) PDE systems (Brun et al., 2020). A generalized THMC representation of bread baking can be expressed through a multiphysics PDE system:

$$\left\{ \begin{array}{l} \text{(Thermal)} \quad \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) - \lambda \frac{\partial W}{\partial t} + Q_{react}, \\ \text{(Hydric)} \quad \frac{\partial W}{\partial t} = \nabla \cdot (D_{\omega} \nabla W), \\ \text{(Chemical)} \quad \frac{\partial C}{\partial t} = D_c \nabla^2 C + R(T, C), \\ \text{(Mechanical)} \quad \nabla \cdot \sigma = 0. \end{array} \right.$$

This unified formulation clarifies how temperature, moisture, chemical reactions, and structural deformation jointly influence the baking process.

PDE-Based Research Gaps

Despite notable advances, several mathematically significant gaps remain:

Gap 1: Lack of Stochastic PDE Modelling

Most models are deterministic, yet bread properties (e.g., pore distribution) are inherently random. Opportunities include stochastic heat equations, random-field diffusion, and probabilistic geometric PDEs.

Gap 2: Insufficient Numerical Stability and Error Analysis

Few studies analyze numerical stability, convergence, or error bounds. This is a major opening for mathematically rigorous contributions. Another gap concerns the limited treatment of moving-boundary phenomena, despite their relevance to crust formation.

Gap 3: Absence of Full 4D (3D + Time) Moving-Boundary PDE Models

Current dough expansion models are simplistic. Promising avenues include dynamic-domain PDEs, level-set geometry tracking, and phase-field PDEs.

Gap 4: Lack of PDE-Based Inverse Problems

Parameter estimation (e.g., thermal diffusivity) rarely uses inverse PDE frameworks, despite high practical relevance.

Gap 5: No PDE-Constrained Optimization Frameworks

Optimal baking profiles could be achieved using optimal control, adjoint PDEs, or PDE-constrained optimization yet this research is nearly nonexistent.

Mathematical Directions for Future Research

Based on these gaps, strong mathematical research directions include: 1) stochastic PDE modelling for structural uncertainty, 2) fully coupled 3D multiphysics PDEs with moving boundaries, 3) integration of machine learning with PDE solvers (e.g., PINNs), 4) PDE-constrained optimization for ideal temperature schedules, 4) theoretical PDE analysis (existence, uniqueness, regularity) applied to baking. Overall, mathematics provides the deepest and most rigorous framework for understanding bread baking, positioning the field as a rich and challenging domain for advanced PDE research.

The results of this review clearly demonstrate that future advancements in bread-baking science will depend on expanding the mathematical sophistication of PDE models. This includes integrating stochastic PDEs for uncertainty quantification, employing inverse PDE frameworks for parameter estimation, and developing PDE-constrained optimization for real-time control of baking processes (Liu & Gerstoft, 2023). These directions underscore the essential role of mathematics in shaping the next generation of predictive food-processing technologies (Cuomo et al., 2022).

5. CONCLUSION

This study provides a comprehensive bibliometric and conceptual analysis of how partial differential equations (PDEs) underpin the scientific understanding of bread baking. By mapping the intellectual structure of the field, the study identifies core themes such as heat transfer, moisture migration, crust formation. This study contributes to the existing knowledge base by clarifying the evolution, conceptual density, and methodological direction of PDE applications in bread science. Future research could expand this framework by integrating machine learning with PDE solvers, improving parameter estimation through real-time sensing, and exploring generalizability across other porous food matrices, thereby advancing both mathematical modeling and food process optimization.

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